7 MAXWELL'S FIELD EQUATIONS AND PROPAGATION OF PLANE ELECTROMAGNETIC WAVES

In chapter sixth, we have dealt with time dependent electric and magnetic fields. We want to take up their discussion again in context of Maxwell field equations. Gauss law $\vec{\nabla}.\mathbf{D} = \rho$ or $\vec{\nabla}.\mathbf{E} = \frac{\rho}{\varepsilon_0}$ derived for time-independent field **E** in dielectrics also holds good for the time-dependent electric field (it forms Maxwell's first equation). Similarly, the equation $\vec{\nabla}.\mathbf{B} = 0$ which was derived from Biot-Savart law for steady currents, also holds good for time-dependent magnetic flux density vector **B** generated by time varying currents (it forms Maxwell's second equation). The third relation $\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ is the Faraday's law involving time-dependent magnetic fields (it forms Maxwell's third equation). Thus so far stated these three relations hold good for time-dependent fields. The difficulty arises with the Ampere's law $\vec{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$ which, as such, can not be accomodated for time-dependent field as explained below :

As the old rule states that divergence of curl is always zero and is very well obeyed by the third relation *i.e.*,

$$\vec{\nabla}.(\vec{\nabla} \times \mathbf{E}) = \vec{\nabla}.\left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}.\mathbf{B}) = 0$$

but not so in case of fourth relation *i.e.*,

$$\vec{\nabla}.(\vec{\nabla} \times \mathbf{B}) = \mu_0 (\vec{\nabla}.\mathbf{J})$$

in which, though $\vec{\nabla}.\mathbf{J} = 0$ in magnetostatics *i.e.*, for *steady* current but not beyond magnetostatics. Beyond magnetostatics Ampere's law cannot be right as we derive it, afterall, from Biot-Savart law. It means divergence of curl is not zero. It is this flaw which was removed by Maxwell by adding a term $\mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (or $\mu_0 \frac{\partial \mathbf{D}}{\partial t}$ where **D** is displacement vector and $\frac{\partial \mathbf{D}}{\partial t}$ is displacement current density) to $\mu_0\mathbf{J}$ and this modification does not effect the case of magnetostatics where **E** is constant and so $\frac{\partial \mathbf{E}}{\partial t} = 0$ (that is, we still have $\vec{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$).

In next article 7.1, we shall derive equation of continuity $\vec{\nabla}.\mathbf{J} = -\frac{\partial \rho}{\partial t}$ that clearly predicts that for stationary currents (charge density ρ is constant) $\vec{\nabla}.\mathbf{J} = 0$ but for time-dependent fields $\frac{\partial \rho}{\partial t} \neq 0$ and so $\vec{\nabla}.\mathbf{J} \neq 0$. Thus Maxwell suggested that for time dependent fields the definition of total current density is incomplete and he added a term $\mathbf{J}'$ to it, to be recognised as displacement current density $\left(\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t}\right)^*$. Therefore, before arriving at Maxwell's field equations, we shall briefly discuss the equation of continuity and the displacement current.

*Refer to solved example-4.

7·0. EQUATION OF CONTINUITY

According of the principle of conservation of charge the net amount of charge in an isolated system remains constant. For generality let us assume that charge density is a function of time. Then the principle of conservation of charge can be stated as follows :

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in a manner that the time rate of decrease of charge within the volume equals the net rate of flow of charge out of the volume. This statement of conservation of charge can be expressed by the equation of continuity which we shall derive here.

As we are now dealing with charges in motion, let us consider that charge density, ρ, is a function of time. The transport of charge constitutes the current *i.e.*,

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_V \rho\, dV, \qquad \dots (1)$$

where we have considered that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area (normal to the direction of charge flow) of a surface in unit time is defined as the current density **J**. According to all experiments to date charge in a closed system is always *conserved*. Therefore if a net amount of current is flowing outward a closed surface, the charge contained within that volume should decrease at the rate,

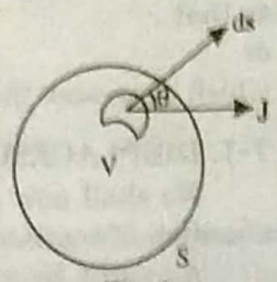

Fig. 1.

$$-\frac{dq}{dt} = I \qquad \dots (2)$$

where I is the total current flowing through surface **S**. If **J** is the current density, then by definition, total current I will be

$$I = \oint_S \mathbf{J}.d\mathbf{S} \qquad \dots (3)$$

From eqs. (2) and (3), we get

$$\oint_S \mathbf{J}.d\mathbf{S} = -\frac{dq}{dt}$$

$$= -\frac{d}{dt}\int_V \rho\, dV, \qquad \dots (4)$$

using eq. (1).

Because it is ρ which is changing with time, we can write

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV,$$

so that eq. (4), becomes

$$\oint_S \mathbf{J}.d\mathbf{S} = -\int_V \frac{\partial \rho}{\partial t}\, dV. \qquad \dots (5)$$

From divergence theorem, we have

$$\oint_S \mathbf{J}.d\mathbf{S} = \int_V (\text{div } \mathbf{J})\, dV,$$

so that eq. (5), becomes

$$\int_V (\text{div } \mathbf{J})\, dV = -\int_V \frac{\partial \rho}{\partial t}\, dV$$

or
$$\int_V \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0 \qquad \text{...(6)}$$

Since eq. (6) holds for any arbitrary volume, we can put integrand equal to zero, *i.e.*,

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \qquad \text{...(7)}$$

and is referred to as the *equation of continuity*. It is the mathematical expression for the conservation of charge. It states that the *'total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that charge can not be created or destroyed, i.e. no sources and sinks are present in that volume'*. In case of *stationary currents*, charge density at any point within the region remains constant.

$$\frac{\partial \rho}{\partial t} = 0,$$

so that
$$\text{div } \mathbf{J} = 0$$
or
$$\vec{\nabla} \cdot \mathbf{J} = 0, \qquad \text{...(8)}$$

which expresses the fact that there is no net outward flux of current density **J**.

7·1. DISPLACEMENT CURRENT

We shall now see how Maxwell changed the definition of total current density to adapt the equation of continuity to time dependent fields.

Ampere's circuital law is

$$\oint_S \mathbf{B}.d\mathbf{l} = \mu_0 I$$

or
$$\oint_S \mathbf{H}.d\mathbf{l} = I = \int_S \mathbf{J}.d\mathbf{S}$$

Changing line integral into surface integral by Stoke's theorem,

$$\int_S \text{Curl } \mathbf{H}.d\mathbf{S} = \int_S \mathbf{J}.d\mathbf{S}$$

or
$$\text{Curl } \mathbf{H} = \mathbf{J} \qquad \text{...(1)}$$

Let us put it in equation of continuity which is

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

we get
$$\text{div(curl } \mathbf{H}) = -\frac{\partial \rho}{\partial t}$$
$$0 = -\frac{\partial \rho}{\partial t}.$$

That is, eq. (1) leads to steady state conditions in which charge density is not changing. Therefore, for time dependent (changing) fields eq. (1) should be modified. Maxwell suggested that the definition of total current density is incomplete and advised to add something to **J**. Let it be **J′**. Then eq. (1) becomes

$$\text{Curl } \mathbf{H} = (\mathbf{J} + \mathbf{J}') \qquad \text{...(2)}$$

In order to identify **J′**, we take divergence of eq. (2). That is,

$$\text{div(curl } \mathbf{H}) = \text{div } (\mathbf{J} + \mathbf{J}')$$
$$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}'$$

or
$$\text{div } \mathbf{J}' = -\text{ div } \mathbf{J} = \frac{\partial \rho}{\partial t} \qquad \text{...(3)}$$

We know that

$$\rho = \vec{\nabla}.\mathbf{D}$$

so that eq. (3) becomes

$$\text{div } \mathbf{J}' = \frac{\partial}{\partial t}(\vec{\nabla}.\mathbf{D})$$

$$= \vec{\nabla}.\frac{\partial \mathbf{D}}{\partial t}$$

$$= \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

or

$$\text{div}\left(\mathbf{J}' - \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \quad \text{... (4)}$$

Since eq. (4) is true for any arbitrary volume, we can put

$$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t} \quad \text{... (5)}$$

Therefore the modified form of the ampere's law is

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{... (6)}$$

for a good conductor. Therefore for $f < 10^{15}$ c/s. $J' \ll J$ and is negligible. We see that ratio of conduction current to the displacement current is very high in the entire frequency range.

7·2. THE MAXWELL'S EQUATIONS (Differential Form) :

We shall now state in differential form, the four equations of Maxwell :

(i) $\vec{\nabla}.\mathbf{D} = \rho$ → obtained by the application of Gauss theorem to electrostatics. **D** is the electric displacement in coulomb/meter^2 and ρ is the free charge density in coulomb/meter^3.

(ii) $\vec{\nabla}.\mathbf{B} = 0$ → obtained by the application of Gauss theorem to magnetic field. **B** is the magnetic induction in weber/meter^2.

(iii) $\vec{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ → obtained by Faraday's and Lenz's law of electromagnetic induction. **E** is the electric intensity in volt/meter.

(iv) $\vec{\nabla} \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$ → obtained by Maxwell's modification of Ampere's law in a circuital form for magnetic field accompanying an electric current. **H** is magnetic field intensity in ampere/meter. **J** is the current density in ampere/meter^2.

(A) DERIVATION OF MAXWELL'S EQUATIONS :

• **(1)** $\boxed{\text{div } \mathbf{D} = \rho}$

Let us consider a surface S bounding a volume V in a dielectric medium. Referring to Gauss' theorem, the integral $\int \mathbf{E}.d\mathbf{S}$ of the normal component of **E** over any closed surface is equal to the total charge enclosed within the surface. In a dielectric medium, as we have mentioned in the separate chapter on dielectrics, the total charge must include both the *free* and the *polarisation charges* or in other words total charge density at a point in a small volume element dV would be $(\rho + \rho_p)$ where polarisation charge density $\rho_p = -\text{ div } \mathbf{P}$ and ρ is the free charge density at a point in a small volume element dV. Thus total charge density at that point will be, $\rho - (\text{div } \mathbf{P})$. Then Gauss law can be expressed as follows :

$$\int_S \mathbf{E}.d\mathbf{S} = \int_V \text{div } \mathbf{E}.dV = \int \frac{1}{\varepsilon_0} (\rho - \text{div } \mathbf{P})\, dV$$

or
$$\int \text{div } (\varepsilon_0 \mathbf{E} + \mathbf{P})\, dV = \int \rho\, dV$$

The quantity $(\varepsilon_0 \mathbf{E} + \mathbf{P})$ is denoted by a quantity **D**, called the *electric displacement*, so that

$$\int \text{div } \mathbf{D}\, dV = \int \rho\, dV$$

or
$$\int (\text{div } \mathbf{D} - \rho)\, dV = 0$$

Since the equation is true for all volumes, the integrand in this equation must vanish, *i.e.*,

$$\text{div } \mathbf{D} = \rho.$$

When the medium is isotropic the three vectors **D**, **E**, **P** are in the same direction and for small field, **D** is proportional to **E**, that is

$$\mathbf{D} = \varepsilon \mathbf{E},$$

where ε is called dielectric constant of the medium.

• (2) $\boxed{\text{div } \mathbf{B} = 0}$

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary close surface is exactly the same as leaving it. It means the flux of magnetic induction **B** across any closed surface is always zero, *i.e.*

$$\int \mathbf{B}.d\mathbf{S} = 0.$$

Transforming the surface integral into volume integral, we have

$$\int \text{div}\, \mathbf{B}\, dV = 0.$$

The integrand should vanish for the surface boundary as the volume is arbitrary, *i.e.*,

$$\text{div}\, \mathbf{B} = 0.$$

• (3) $\boxed{\text{Curl}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$

By Faraday law, we know that e.m.f. induced in a closed loop is given by

$$e = -\frac{\partial \phi}{\partial t} = -\int \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S},$$

since the flux $\phi = \int_S \mathbf{B}.d\mathbf{S}$ where S is any surface having the loop as boundary.

e.m.f., e, can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus

$$e = \oint \mathbf{E}.d\mathbf{l}$$

where **E** is the intensity of the electric field associated with induced e.m.f.

Therefore, equating above two equations, we get

$$\oint \mathbf{E}.d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}.$$

Applying Stoke's theorem, the line integral can be transformed into surface integral, *i.e.*,

$$\int_S (\vec{\nabla} \times \mathbf{E}).d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}.$$

This equation must be true for any surface whether small or large in the field. Therefore the two vectors in the integrands must be equal at every point, *i.e.*,

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

or

$$\text{Curl}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

• (4) $\boxed{\text{Curl}\ \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$

Ampere's law in the circuital form gives this equation. According to this law, the work done in carrying a unit magnetic pole once round a closed arbitrary path linked with the current is expressed as

$$\oint \mathbf{H}.d\mathbf{l} = I.$$

or

$$= \int \mathbf{J}.d\mathbf{S},$$

where the integral on the right is taken over the surface through which the charge flow corresponding to the current I takes place.

Now changing the line integral into surface integral by Stoke's theorem,

$$\int_S \text{Curl } \mathbf{H}.d\mathbf{S} = \int \mathbf{J}.d\mathbf{S}.$$

$$\text{Curl } \mathbf{H} = \mathbf{J}.$$

The above relation, derived on the basis of Ampere's law, stands only for steady closed current but for the changing electric fields, the current density should be modified. The difficulty with above equation is that if we take divergence of above equation then

$$\text{div (curl } \mathbf{H}) = 0,$$

or div $\mathbf{J} = 0$ which conflicts with the equation of continuity div $\mathbf{J} = -\partial\rho/\partial t$. Therefore Maxwell realised that the definition of total current density is incomplete and suggested to add something to $\mathbf{J}$, *i.e.*,

$$\text{Curl } \mathbf{H} = (\mathbf{J} + \mathbf{J}').$$

Now taking divergence of above equation, we get

$$\text{div (curl } \mathbf{H}) = (\text{div } \mathbf{J} + \text{div } \mathbf{J}'),$$

or $$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}'$$

or $$\text{div } \mathbf{J}' = -\text{ div } \mathbf{J} = +\frac{\partial \rho}{\partial t}.$$

We know that $$\rho = \vec{\nabla}.\mathbf{D}.$$

Putting this value in the expression for div $\mathbf{J}'$, we get

$$\text{div } \mathbf{J}' = \frac{\partial}{\partial t}(\vec{\nabla}.\mathbf{D})$$

$$\vec{\nabla}.\mathbf{J}' = \vec{\nabla}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

Hence $$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t}.$$

Therefore the Maxwell's fourth relation can be written as

$$\text{Curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

It is quite obvious from the relation for $\mathbf{J}'$ that the term $\partial\mathbf{D}/\partial t$ on right hand side arises when the electric displacement $\mathbf{D}$ changes with time and is, therefore, termed as *displacement current density*. According to Maxwell, it is just as effective as $\mathbf{J}$ in producing magnetic field.

(B) WORD STATEMENT OF THE FIELD EQUATIONS (Maxwell's equation in integral form) :

The significance of the field equation is readily obtained from their mathematical statement in integral form.

- **(1) For equation** $\vec{\nabla}.\mathbf{D} = \rho$

Integrating this equation, over a volume V, we arrive at

$$\int_V \vec{\nabla}.\mathbf{D}\, dV = \int_V \rho\, dV.$$

But from Gauss theorem, we get

$$\int_S \mathbf{D}.d\mathbf{S} = \int_V \rho\, dV = q,$$

where q is the net charge contained in volume V. S is the surface bounding volume V.

Therefore this Maxwell's equation signifies that :

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

- **(2) For equation** $\vec{\nabla}.\mathbf{B} = 0.$

Exactly in a manner adoped above, we can arrive at

$$\int_S \mathbf{B}.d\mathbf{S} = 0$$

which signifies that :

The total outward flux of magnetic induction **B** *through any closed surface S is equal to zero.*

- **(3) For equation** $\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

Integrating this equation over a surface S, bounded by a curve we arrive at

$$\int_S (\vec{\nabla} \times \mathbf{E}) \,.\, d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \,.\, d\mathbf{S}.$$

Converting the surface integral of left hand side into line integral by Stoke's theorem, we get

$$\oint \mathbf{E} \,.\, dl = -\frac{\partial}{\partial t} \int_S \mathbf{B}.d\mathbf{S}$$

which signifies that :

The electromotive force around a closed path is equal to the time derivative of the magnetic flux through any surface bound by the path.

- **(4) For equation** $\vec{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$.

which can be written in integral form as

$$\oint \mathbf{H}.dl = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) . d\mathbf{S},$$

on proceeding exactly in the way adopted in point (3). This form signifies that :

The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by the path.

The correspondence of **B** and **H** with **E** and **D** through the Maxwell's curl equations (3) and (4) implies that time varying electric and magnetic fields in empty space are independent *i.e.*, a changing electric field being able to generate a magnetic field and *vice-versa*. From this we draw the inference that a *time-changing electromagnetic field would propagate energy through empty space with the velocity of light* and *further, that the light is electromagnetic in nature.*